A PARAMETRIC FAMILY OF QUINTIC THUE EQUATIONS

ISTVÁN GAÁL AND GÜNTER LETTL

ABSTRACT. For an integral parameter $t \in \mathbb{Z}$ we investigate the family of Thue equations

$$F(x, y) = x^5 + (t - 1)^2 x^4 y - (2t^3 + 4t + 4)x^3 y^2$$
$$+ (t^4 + t^3 + 2t^2 + 4t - 3)x^2 y^3 + (t^3 + t^2 + 5t + 3)xy^4 + y^5 = \pm 1,$$

originating from Emma Lehmer's family of quintic fields, and show that for $|t| \geq 3.28 \cdot 10^{15}$ the only solutions are the trivial ones with $x = 0$ or $y = 0$. Our arguments contain some new ideas in comparison with the standard methods for Thue families, which gives this family a special interest.

1. INTRODUCTION

For $t \in \mathbb{Z}$ let us define the polynomial

$$f_t(x) = x^5 + (t - 1)^2 x^4 - (2t^3 + 4t + 4)x^3$$
$$+ (t^4 + t^3 + 2t^2 + 4t - 3)x^2 + (t^3 + t^2 + 5t + 3)x + 1.$$

This family of quintic polynomials was first considered by Emma Lehmer (cf. [6]). Note that instead of the original parameter n we use the parameter $t = n + 1$, which fits our arguments better (cf. [11, p. 548]). The corresponding parametric family of totally real cyclic quintic fields $K_t = \mathbb{Q}(\vartheta_t)$, ϑ_t a root of f_t, was also investigated by Schoof and Washington [11] and Darmon [2] for prime conductors $m_0 = t^4 + t^3 + 6t^2 + 6t + 11$.

In a recent paper Gaál and Pohst [3] showed that any four distinct roots of f_t form a fundamental system of units in K_t for any conductor m_0, constructed explicitly an integral basis of K_t for those m_0 that are square free apart from 5, and studied the problem of power integral bases.

In the present paper we will investigate the corresponding family of quintic Thue equations

$$(1) \quad F(x, y) = x^5 + (t - 1)^2 x^4 y - (2t^3 + 4t + 4)x^3 y^2$$
$$+ (t^4 + t^3 + 2t^2 + 4t - 3)x^2 y^3 + (t^3 + t^2 + 5t + 3)xy^4 + y^5 = \pm 1.$$

The first infinite parametric family of Thue equations was a cubic one, studied by Thomas [12]. He proved that if the parameter is large enough, the equation has only the trivial solutions. His ideas were extended to several other families of cubic

and quartic Thue equations in Thomas [13], Pethő [9], Mignotte, Pethő and Roth [8], Lettl and Pethő [7], Pethő and Tichy [10]; for a survey see [5]. Heuberger [4] was the first who obtained results on a quintic family.

The above quoted papers usually apply some standard techniques for Thue equations and Baker's method to derive an upper bound for the unknown exponents of the fundamental units (corresponding to a non-trivial solution) in terms of the parameter. If the rank of the unit group is small, a difference in the size of the logarithms of the fundamental units implies a difference in the order of magnitude of the exponents. This allows us to derive a lower bound for the largest exponent, contradicting the upper bound and thus proving the non-existence of non-trivial solutions if the parameter is large enough.

Investigating equation (1), we have to handle a unit group of rank 4, and it turns out that the exponents have the same order of magnitude. So we had to design a new method to overcome this problem. Using asymptotic expansions of the expressions involved, we construct non-vanishing linear combinations of the exponents, which are small. These allow us to derive lower bounds for $|y|$ and the largest exponent, and again one obtains the desired contradiction for sufficiently large parameters. This method can be applied for families of Thue equations with high unit rank, as long as the coefficients of the asymptotic expansions of the roots with respect to the parameter are rational.

The main result of this paper is the following.

Theorem. *For $|t| \geq 3.28 \cdot 10^{15}$ the only integral solutions of equation (1) are the trivial ones,*

$$(x, y) = (\pm 1, 0), (0, \pm 1).$$

We remark that for the calculations involved in our estimates we extensively used Maple.

2. Elementary estimates

For $t \in \mathbb{Z}$, let α be a root of the polynomial $f_t(x) = F(x, 1)$. The field $K = \mathbb{Q}(\alpha)$ is a totally real cyclic field, the Galois group of which is generated by the automorphism

$$\varphi : \alpha \longmapsto \alpha' = \frac{(t+1) + (t-1)\alpha - \alpha^2}{1 + (t+1)\alpha}.$$

We denote the roots of $f_t(x)$ by $\alpha_1, \ldots, \alpha_5$, such that α_1 is the smallest one and $\varphi(\alpha_i) = \alpha_{i+1}$ for $i = 1, \ldots, 5$ (note that here and in the following the indices of the α_i are to be taken mod(5)). According to [3], any four distinct roots of $f_t(x)$ form a system of fundamental units in K.

From the asymptotic expansions of the roots we obtain that for $|t| \geq 100$ one has

$$\alpha_1 = -t^2 - 2 - \frac{1 + \delta_1}{t}, \qquad \alpha_2 = t + \frac{1}{t} - \frac{1}{t^3} - \frac{1 + \delta_2}{t^4},$$

$$(2) \qquad \alpha_3 = -\frac{1}{t^3} + \frac{1}{t^4} + \frac{3}{t^5} - \frac{4 + \delta_3}{t^6}, \qquad \alpha_4 = t + 1 + \frac{1}{t} - \frac{1}{t^3} - \frac{1 + \delta_4}{t^4},$$

$$\alpha_5 = -\frac{1}{t} + \frac{1 + \delta_5}{t^4},$$

with some $|\delta_i| < 0.1$ $(i = 1, \ldots, 5)$. This can be checked by substituting these expressions with $\delta_i = \pm 0.1$ into $f_t(x)$ and observing that $f_t(x)$ changes its sign between these two arguments. Let us remark that for $t \geq 0$ one can show that $\alpha_1 < \alpha_5 < \alpha_3 < \alpha_2 < \alpha_4$, whereas for $t < 0$ one has $\alpha_1 < \alpha_2 < \alpha_4 < \alpha_3 < \alpha_5$.

If $y = 0$, (1) has only the trivial solutions $(\pm 1, 0)$. So let us assume for the following that $|t| \geq 100$ and that $(x, y) \in \mathbb{Z}^2$ is a solution of (1) with $|y| \geq 1$. Put

$$\beta_j = x - \alpha_j y \quad \text{for} \quad j = 1, \ldots 5,$$

and define i to be that index with $|\beta_i| = \min_{1 \leq j \leq 5} |\beta_j|$. Then for $m \neq i$ we obtain

$$(3) \qquad \left| \alpha_m - \frac{x}{y} \right| < |\alpha_m - \tilde{\alpha}_{i,m}|,$$

where $\tilde{\alpha}_{i,m}$ is the arithmetic mean of α_i and its neighbouring root between α_i and α_m. (This may be different for $t \geq 0$ and $t < 0$. Indeed, if e.g. $i = 4$ and $t \geq 0$ then $\tilde{\alpha}_{4,m} = (\alpha_2 + \alpha_4)/2$ for all $m \neq 4$, whereas for $t < 0$ one has $\tilde{\alpha}_{4,1} = \tilde{\alpha}_{4,2} = (\alpha_2 + \alpha_4)/2$ and $\tilde{\alpha}_{4,3} = \tilde{\alpha}_{4,5} = (\alpha_3 + \alpha_4)/2$.) Using (3), we deduce that

$$\left| \alpha_i - \frac{x}{y} \right| = \frac{1}{|y|^5} \cdot \frac{1}{\prod_{m \neq i} \left| \alpha_m - \frac{x}{y} \right|} \leq \frac{1}{|y|^5} \cdot \frac{1}{\prod_{m \neq i} |\alpha_m - \tilde{\alpha}_{i,m}|},$$

and by estimates (2) for the roots we obtain, for all $i = 1, \ldots, 5$,

$$(4) \qquad \left| \alpha_i - \frac{x}{y} \right| < \frac{8.06}{|y|^5 |t|^3}.$$

Since $|t| \geq 100$, this implies $|\alpha_i - x/y| < 1/(2 |y|^2)$; thus x/y is a convergent arising from the continued fraction expansion of α_i. Calculating the continued fraction expansion of the roots, we obtain for $t \geq 5$

$$\alpha_1 = \langle -t^2 - 3; 1, t - 1, [t/3], \ldots \rangle,$$
$$\alpha_2 = \langle t; t, t - 1, \ldots \rangle,$$
$$\alpha_3 = \langle -1; 1, t^3 + t^2 + 4t + 2, \ldots \rangle,$$
$$\alpha_4 = \langle t + 1; t, t - 1, \ldots \rangle,$$
$$\alpha_5 = \langle -1; 1, t - 1, t^2 - t + 4, \ldots \rangle,$$

and for $t \leq -8$

$$\alpha_1 = \langle -t^2 - 2; |t|, [(|t| - 1)/3], \ldots \rangle,$$
$$\alpha_2 = \langle t - 1; 1, |t| - 1, |t| + 1, \ldots \rangle,$$
$$\alpha_3 = \langle 0; -t^3 - t^2 - 4t - 3, [|t|/2] - 1, \ldots \rangle,$$
$$\alpha_4 = \langle t; 1, |t| - 1, |t| + 1, \ldots \rangle,$$
$$\alpha_5 = \langle 0; |t| - 1, 1, t^2 + |t| + 4, \ldots \rangle,$$

where $[x]$ denotes the largest integer not exceeding x. To get the continued fraction expansion of the roots we had Maple calculate them for $-20 \leq t \leq 20$, from which we guessed the general shape of the partial quotients. For the proof of these expansions let us note that the partial quotients a_j for all $0 \leq j \leq m$ of the continued fraction expansion of some real number $\alpha = [a_0; a_1, a_2, \ldots]$ are determined by the property that α lies between $[a_0; a_1, \ldots, a_m]$ and $[a_0; a_1, \ldots, a_m + 1]$. Calculating the above expressions with (an upper bound for) the last indicated partial quotient

a_m and (a lower bound for) $a_m + 1$, resp., one verifies that $f_t(x)$ changes its sign between these two values for all t which are to be considered.

From the above continued fractions we can calculate all convergents of the roots with denominators less than $t^2 + 1$ (in the case of α_1 one also has to consider some possible small values for the next two partial quotients which are not indicated in the above expansions). Substituting the numerators and denominators of these convergents for x, y into (1), we do not get any solutions except the trivial one $(0, \pm 1)$. For this reason we conclude that

$$(5) \qquad\qquad |y| > t^2 \,.$$

3. BAKER'S METHOD

Let j, k be such that i, j, k are distinct indices among $1, \dots, 5$. From Siegel's identity

$$(\alpha_i - \alpha_j)\beta_k + (\alpha_j - \alpha_k)\beta_i + (\alpha_k - \alpha_i)\beta_j = 0$$

and using (4), we obtain

$$\left| \left| \frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k} \cdot \frac{\beta_k}{\beta_j} \right| - 1 \right| \leq \left| \frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k} \cdot \frac{\beta_k}{\beta_j} - 1 \right| = \left| \frac{\alpha_k - \alpha_j}{\alpha_i - \alpha_k} \cdot \frac{\beta_i}{\beta_j} \right|$$

$$\leq \left| \frac{\alpha_k - \alpha_j}{\alpha_i - \alpha_k} \right| \cdot \left| \frac{\alpha_i - \frac{x}{y}}{\alpha_j - \frac{x}{y}} \right| < \left| \frac{\alpha_k - \alpha_j}{\alpha_i - \alpha_k} \right| \cdot \frac{8.06\, |y|^{-5} |t|^{-3}}{|\alpha_j - \alpha_i| - 8.06\, |t|^{-3}} \,.$$

In order to get appropriate estimates we choose $(i, j, k) = (1, 3, 5), (2, 5, 3), (3, 1, 4),$ $(4, 1, 3), (5, 2, 4)$ in the five cases that are to be considered, depending on the value of i. By (2) one obtains that for all t with $|t| \geq 100$

$$\left| \frac{\alpha_k - \alpha_j}{\alpha_i - \alpha_k} \right| \cdot \frac{8.06\, |y|^{-5} |t|^{-3}}{|\alpha_j - \alpha_i| - 8.06\, |t|^{-3}} < \frac{8.1}{|y|^5 |t|^4} \,.$$

Since for any real $z \geq 0.2032$ the inequality $|\log(z)| < 2\,|z - 1|$ holds, we finally conclude from these estimates that for each $i \in \{1, \dots, 5\}$

$$(6) \qquad \Lambda_i := \left| \log \left| \frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k} \cdot \frac{\beta_k}{\beta_j} \right| \right| \leq 2 \left| \left| \frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k} \cdot \frac{\beta_k}{\beta_j} \right| - 1 \right| < \frac{16.2}{|y|^5 |t|^4} \,.$$

Next we are going to apply Baker's method to get a lower estimate for Λ_i. We take $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as a system of fundamental units. Then there are $u_1, \dots, u_4 \in \mathbb{Z}$ such that for $m = 1, \dots, 5$ we have

$$(7) \qquad\qquad \beta_m = \alpha_m^{u_1} \, \alpha_{m+1}^{u_2} \, \alpha_{m+2}^{u_3} \, \alpha_{m+3}^{u_4} \,,$$

and so

$$(8)$$

$$\Lambda_i = \left| \log \left| \frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k} \right| + u_1 \log \left| \frac{\alpha_k}{\alpha_j} \right| + u_2 \log \left| \frac{\alpha_{k+1}}{\alpha_{j+1}} \right| + u_3 \log \left| \frac{\alpha_{k+2}}{\alpha_{j+2}} \right| + u_4 \log \left| \frac{\alpha_{k+3}}{\alpha_{j+3}} \right| \right| \,.$$

To obtain good estimates for $\log |\alpha_m|$ and $\log |\alpha_m - \alpha_n|$ we derive from the series expansion of the logarithm the following auxiliar result:

Lemma. *Let* $a_1, a_2, a_3, t \in \mathbb{R}$ *with* $|t| \geq 100$, $|a_1| \leq 1$, $|a_2| \leq 3$, $|a_3| \leq 5$ *and* $-3 < -a_1 a_3 + a_1^2 a_2 < 8$. *Then*

$$\log \left(1 + \frac{a_1}{t} + \frac{a_2}{t^2} + \frac{a_3}{t^3} \right) = \frac{a_1}{t} + \frac{a_2 - a_1^2/2}{t^2} + \frac{a_3 - a_1 a_2 + a_1^3/3 + \tilde{\delta}}{t^3}$$

for some $\tilde{\delta} \in \mathbb{R}$ *with* $|\tilde{\delta}| < 0.1$.

Proof. Putting $c := a_1/t + a_2/t^2 + a_3/t^3$, the above conditions yield $|c| \le 1.0305/|t|$, and by estimating the series expansion of the logarithm we obtain

$$0 \le c - \frac{c^2}{2} + \frac{c^3}{3} - \log(1+c) < \frac{0.003}{|t|^3}.$$

Next one can show that the difference between $c - c^2/2 + c^3/3$ and the expression claimed in the lemma is at most $0.083/|t|^3$. Combining these inequalities, we get the assertion. □

Applying our lemma and using (2), we obtain the following estimates:

$$\log|\alpha_1| = 2\log|t| + \frac{2}{t^2} + \frac{1+\delta_6}{t^3},$$

$$\log|\alpha_2| = \log|t| + \frac{1}{t^2} + \frac{\delta_7}{t^3},$$

(9)
$$\log|\alpha_3| = -3\log|t| - \frac{1}{t} - \frac{7}{2t^2} + \frac{2/3+\delta_8}{t^3},$$

$$\log|\alpha_4| = \log|t| + \frac{1}{t} + \frac{1}{2t^2} + \frac{-2/3+\delta_9}{t^3},$$

$$\log|\alpha_5| = -\log|t| + \frac{-1+\delta_{10}}{t^3},$$

where $|\delta_m| < 0.2$, $m = 6, \dots, 10$, and similarly,

$$\log|\alpha_1 - \alpha_2| = 2\log|t| + \frac{1}{t} + \frac{3}{2t^2} + \frac{1/3+\delta_{11}}{t^3},$$

$$\log|\alpha_1 - \alpha_3| = 2\log|t| + \frac{2}{t^2} + \frac{1+\delta_{12}}{t^3},$$

$$\log|\alpha_1 - \alpha_4| = 2\log|t| + \frac{1}{t} + \frac{5}{2t^2} + \frac{-2/3+\delta_{13}}{t^3},$$

$$\log|\alpha_1 - \alpha_5| = 2\log|t| + \frac{2}{t^2} + \frac{\delta_{14}}{t^3},$$

(10)
$$\log|\alpha_2 - \alpha_3| = \log|t| + \frac{1}{t^2} + \frac{\delta_{15}}{t^3},$$

$$\log|\alpha_2 - \alpha_4| = \frac{\delta_{16}}{t^3},$$

$$\log|\alpha_2 - \alpha_5| = \log|t| + \frac{2}{t^2} + \frac{\delta_{17}}{t^3},$$

$$\log|\alpha_3 - \alpha_4| = \log|t| + \frac{1}{t} + \frac{1}{2t^2} + \frac{-2/3+\delta_{18}}{t^3},$$

$$\log|\alpha_3 - \alpha_5| = -\log|t| - \frac{1}{t^2} + \frac{\delta_{19}}{t^3},$$

$$\log|\alpha_4 - \alpha_5| = \log|t| + \frac{1}{t} + \frac{3}{2t^2} + \frac{-5/3+\delta_{20}}{t^3},$$

where $|\delta_m| < 0.3$, $m = 11, \dots, 20$.

Now we apply the theorem of Baker and Wüstholz [1, p. 20] to the linear form of logarithms (8) with $n = d = 5$ and $U = \max\{|u_1|, \dots, |u_4|\}$. This yields

$$\log|\Lambda_i| > -18 \cdot 6! \, 5^6 160^7 \log 50 \cdot h'\left(\frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k}\right) h'\left(\frac{\alpha_k}{\alpha_j}\right)^4 \log U.$$

Note that the terms of the linear forms are linearly independent over $\mathbb{Q}$, since by our choice of the indices (i, j, k) the numbers $(\alpha_i - \alpha_j)/(\alpha_i - \alpha_k)$ are not units. Using (9) and (10), it is a routine matter to estimate the heights of the algebraic numbers involved, and we obtained

$$h'\left(\frac{\alpha_k}{\alpha_j}\right) = h'\left(\frac{\alpha_1}{\alpha_3}\right) < 1.0005 \log|t| \quad \text{and} \quad h'\left(\frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k}\right) < 0.801 \log|t|,$$

and finally the lower bound

$$(11) \qquad\qquad \Lambda_i > \exp\left(-0.1707 \cdot 10^{25} \left(\log|t|\right)^5 \log U\right),$$

valid for each i.

4. A LOWER BOUND FOR U

Taking logarithms of the absolute values of (7) yields the system of equations

(12)

$$\log|y| + \log\left|\alpha_m - \frac{x}{y}\right| = u_1 \log|\alpha_m| + u_2 \log|\alpha_{m+1}| + u_3 \log|\alpha_{m+2}| + u_4 \log|\alpha_{m+3}|$$

for $m = 1, \ldots, 5$, $m \neq i$. The determinant of this system of linear equations in $u_1, \ldots, u_4$ is, up to sign, just the regulator R_K of the field K, and using (9) we get

$$(13) \qquad \begin{aligned} R_K &= 71(\log|t|)^4 + \frac{56(\log|t|)^3}{t} + \frac{308(\log|t|)^3 + 21(\log|t|)^2 + \delta_{21}}{t^2} \\ &= (71 + \delta_{22})(\log|t|)^4 \end{aligned}$$

with $|\delta_{21}| < 115$ and $|\delta_{22}| < 0.13$.

Note that by (4) and (5) we have

$$\log\left|\alpha_m - \frac{x}{y}\right| = \log\left|\alpha_m - \alpha_i + \frac{\delta_{23}}{t^{13}}\right|,$$

where $|\delta_{23}| < 8.06$. By (2) we obtain for these expressions the same asymptotic expansions as in (10) for $|\alpha_m - \alpha_i|$, again with some $|\delta_m| < 0.3$ for $m = 11, \ldots, 20$.

For each case of $i \in \{1, \ldots, 5\}$ we solve the system of linear equations (12) by using Cramer's rule and obtain

(14)

$$\begin{aligned} R_K u_j =\ &\log|y| \left(d_{j1}(\log|t|)^3 + \frac{d_{j2}(\log|t|)^2}{t} + \frac{d_{j3}(\log|t|)^2 + d_{j4}\log|t| + d_{j5}}{t^2}\right) \\ &+ e_{j1}(\log|t|)^4 + \frac{e_{j2}(\log|t|)^3}{t} + \frac{e_{j3}(\log|t|)^3 + e_{j4}(\log|t|)^2 + e_{j5}}{t^2} \end{aligned}$$

for $j = 1, \ldots, 4$, with rational coefficients $d_{jk}, e_{jk} \in \mathbb{Q}$ for $1 \leq k \leq 4$ and bounds for d_{j5}, e_{j5}. It turns out that in all cases the u_j's have the same order of magnitude. Therefore we look for integral linear combinations $b_0 R_K + b_1 R_K u_1 + \cdots + b_4 R_K u_4$ with $b_j \in \mathbb{Z}$ which are positive and small in such a way that the coefficients of the main terms $\log|y|(\log|t|)^3$, $\log|y|(\log|t|)^2/t$, $(\log|t|)^4$ and $(\log|t|)^3/t$ vanish. This amounts to solving a system of linear Diophantine equations. Then since

$$(15) \qquad\qquad 1 \leq v = b_0 + b_1 u_1 + \cdots + b_4 u_4 \in \mathbb{Z}$$

and

(16)
$$R_K \, v = \log |y| \left(\frac{d_6 (\log |t|)^2 + d_7 \log |t| + d_8}{t^2} \right) + \frac{e_6 (\log |t|)^3 + e_7 (\log |t|)^2 + e_8}{t^2},$$

we can use $R_K \, v \geq R_K$ and (13) to derive a lower bound for $\log |y|$ of the form

(17)
$$\log |y| > c_1 \, t^2 (\log |t|)^2 \,.$$

Explicitly, in the five cases for i we arrived at the following values for the b_j's in (15):

i	b_0	b_1	b_2	b_3	b_4
1	0	10	3	9	-27
2	-22	-65	49	-51	-15
3	1	-1	-21	10	-8
4	6	-6	-5	18	-17
5	1	-5	18	-17	10

and for the coefficients of $R_K \, v$ in (16) and the constant c_1 in (17) we obtained

i	d_6	d_7	$|d_8| <$	e_6	e_7	$|e_8| <$	c_1
1	710	170	136	1349	269	1860	0.094
2	2130	130	591	2769	363	4268	0.032
3	355	180	52	71	0	498	0.178
4	355	-10	114	284	3	989	0.196
5	355	-10	114	-71	-1	670	0.197

From $d_6, e_6 > 0$ in the cases $1 \leq i \leq 4$ we immediately have $R_K \, v > 0$; thus $v \geq 1$. For $i = 5$ we used $\log |y| > 2 \log |t|$ from (5) to show the same.

To improve the value of c_1 in the cases $i = 1, 2$ we looked for other b_j's giving a smaller coefficient d_6 for $R_K \, v$, but no longer requiring the term $(\log |t|)^3 / t$ to vanish, and found

i	b_0	b_1	b_2	b_3	b_4	c_1
1	0	-26	37	-11	-15	0.319
2	-3	-17	10	-6	-5	0.197

With the lower bound (17) for $\log |y|$ obtained in the first step we could show that again $R_K \, v > 0$, and so we get the improved values for c_1 as indicated in the above table.

From the lower bound (17) for $\log |y|$ it is clear that $U = \max_{1 \leq n \leq 4} |u_n| = |u_j|$ for that u_j having the largest coefficient d_{j1} (in absolute value) in its expansion

(14). Using this u_j, (13) and (17), we have

$$(71 + \delta_{22})(\log|t|)^4 U = R_K\,|u_j|$$

$$> c_1\,t^2(\log|t|)^2 \left(|d_{j1}|(\log|t|)^3 - \frac{|d_{j2}|(\log|t|)^2}{|t|} \right.$$

$$\left. - \frac{|d_{j3}|(\log|t|)^2 + |d_{j4}|(\log|t|) + |d_{j5}|}{t^2} \right)$$

$$- |e_{j1}|(\log|t|)^4 - \frac{|e_{j2}|(\log|t|)^3}{|t|} - \frac{|e_{j3}|(\log|t|)^3 + |e_{j4}|(\log|t|) + |e_{j5}|}{t^2},$$

which gives a lower bound for U of the type

$$(18) \qquad\qquad\qquad U > c_2\,t^2 \log|t|$$

with $0.1548 \le c_2 \le 0.285$, depending on the different cases of i.

On the other hand, the inequality

$$(71 + \delta_{22})(\log|t|)^4 U = R_K\,|u_j|$$

$$< \log|y| \left(|d_{j1}|(\log|t|)^3 + \frac{|d_{j2}|(\log|t|)^2}{|t|} + \frac{|d_{j3}|(\log|t|)^2 + |d_{j4}|\log|t| + |d_{j5}|}{t^2} \right)$$

$$+ |e_{j1}|(\log|t|)^4 + \frac{|e_{j2}|(\log|t|)^3}{|t|} + \frac{|e_{j3}|(\log|t|)^3 + |e_{j4}|\log|t| + |e_{j5}|}{t^2}$$

yields

$$(19) \qquad\qquad\qquad c_3\,U \log|t| < \log|y|$$

with $0.686 < c_3 < 1.358$. Combining (6) with (19), we conclude that

$$\Lambda_i < \exp(2.8 - 3.43\,U \log|t| - 4\log|t|) < \exp(-3.43\,U \log|t|).$$

This upper bound together with the lower estimate (11) yields

$$\frac{U}{\log U} < 4.98 \cdot 10^{23}(\log|t|)^4.$$

Using (18), by the monotony of $x/\log x$ we get

$$\frac{0.1548\,t^2 \log|t|}{\log(0.1548\,t^2 \log|t|)} < 4.98 \cdot 10^{23}\,(\log|t|)^4.$$

This inequality holds for $|t| = 3.27 \cdot 10^{15}$ but fails for $|t| = 3.28 \cdot 10^{15}$, which implies that the existence of a non-trivial solution leads to a contradiction if

$$|t| \ge 3.28 \cdot 10^{15}.$$

References

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KOSSUTH LAJOS UNIVERSITY, MATHEMATICAL INSTITUTE, H–4010 DEBRECEN PF.12., HUNGARY
E-mail address: `igaal@math.klte.hu`

KARL-FRANZENS-UNIVERSITÄT GRAZ, INSTITUT FÜR MATHEMATIK, A–8010 GRAZ, HEINRICH-STRASSE 36, AUSTRIA
E-mail address: `guenter.lettl@kfunigraz.ac.at`